Lecture 7
Heaps, Heapsort, Stable sorting, Optimality of Heapsort/Mergesort (revisited)

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Heapsort

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But we can speed this up by using a binary heap.

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  - Most urgent priority may correspond to the lowest key value or to the highest key value, depending on the application.

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- In our examples, items are integers, key is the integer value

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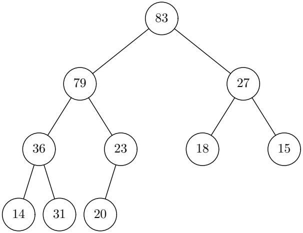

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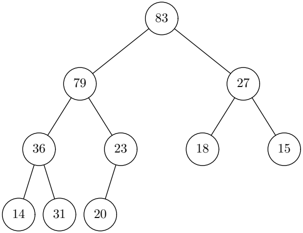

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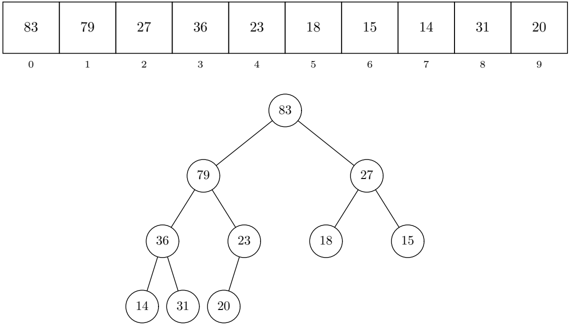

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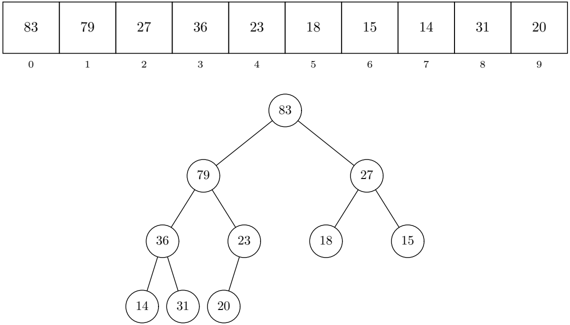

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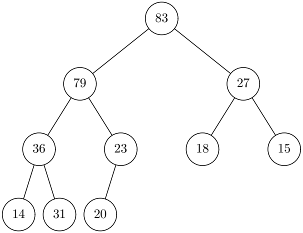

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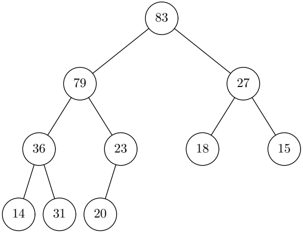

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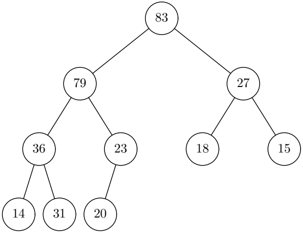

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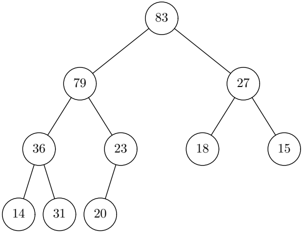

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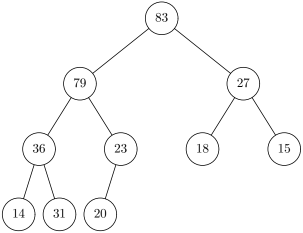

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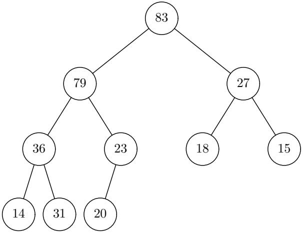

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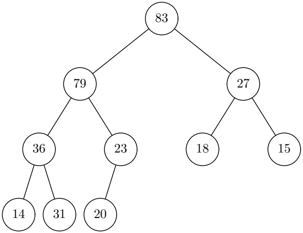

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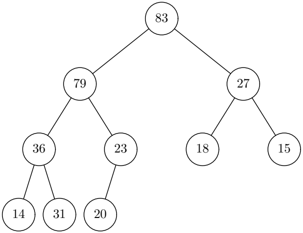

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def FindMax(H):
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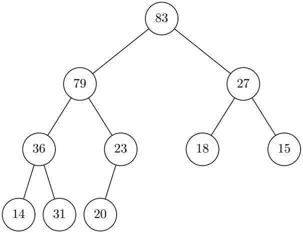

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[GT] calls these "up-heap bubbling" and "down-heap bubbling"

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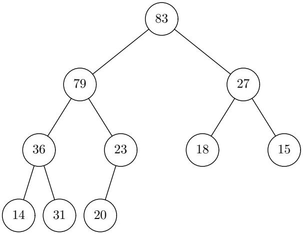

SiftUp: Sift an item up to its correct position

```
def SiftUp(H,i):
    parent = (i-1)/2;
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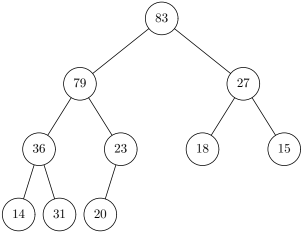

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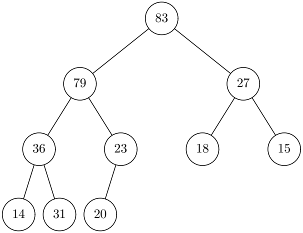
Analysis: at most 1 comparison at each level, so total time is $O(\log n)$



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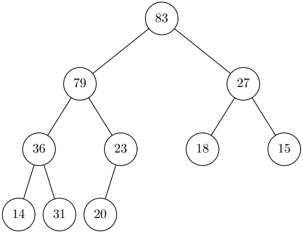

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def SiftDown(H,i):
    n = H.size // number of item in heap
    left = 2i+1; right = 2i+2
    if (right < n) and (H[right].key > H[left].key)
        largerChild = right
    else largerChild = left
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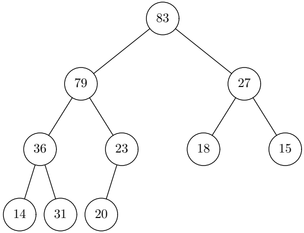

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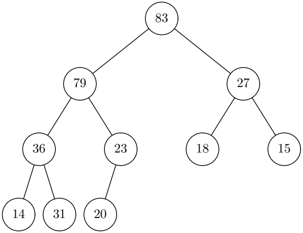
Analysis: at most 2 comparisons at each level, so total time is $O(\log n)$



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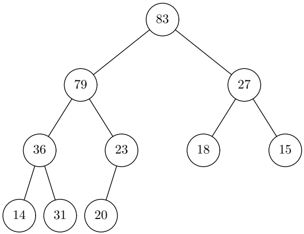

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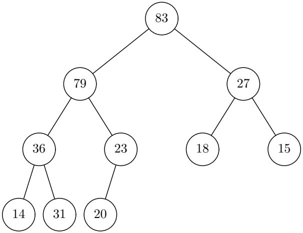

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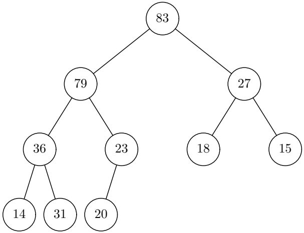

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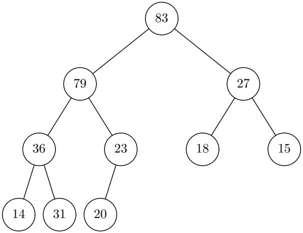

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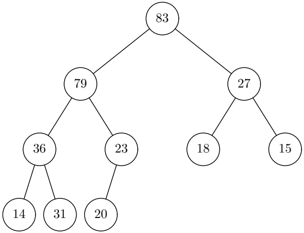

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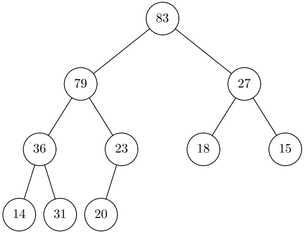

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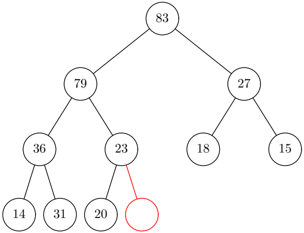

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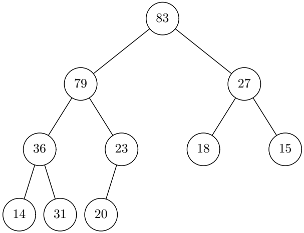

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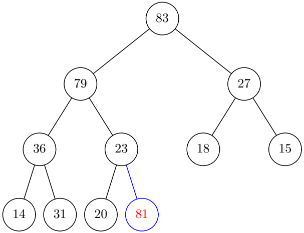

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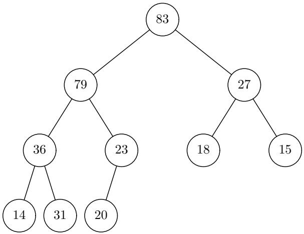

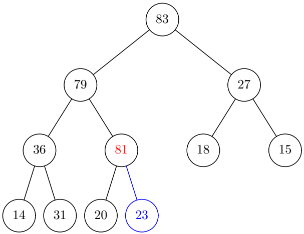

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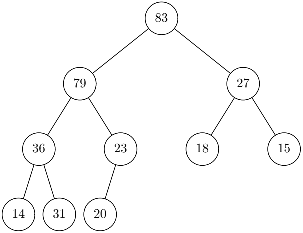

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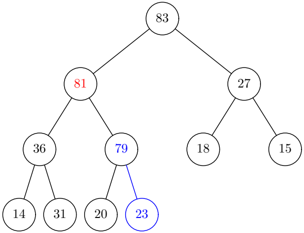

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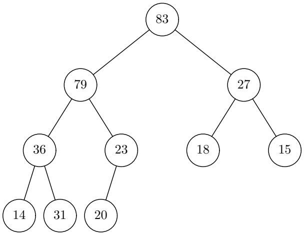

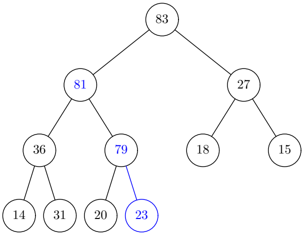

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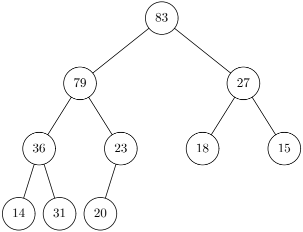

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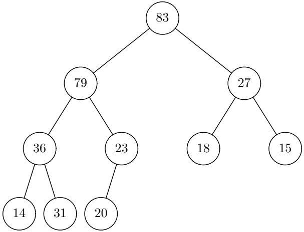

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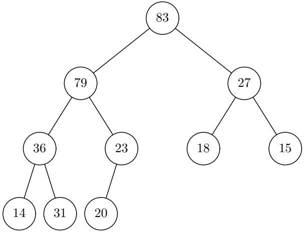

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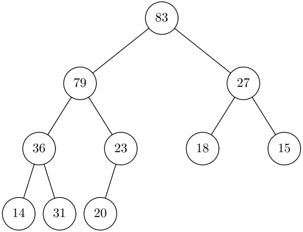

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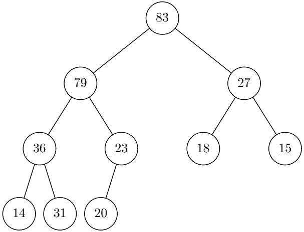

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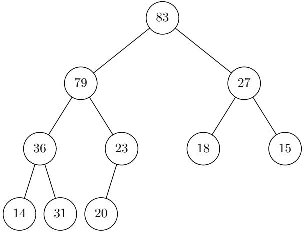

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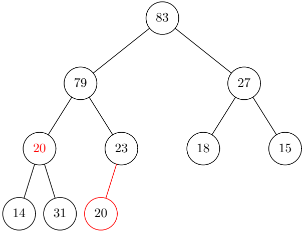

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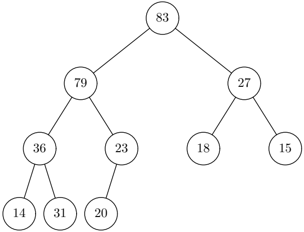

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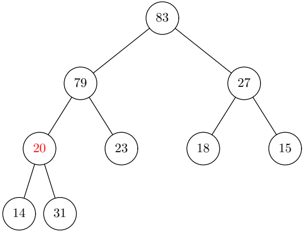

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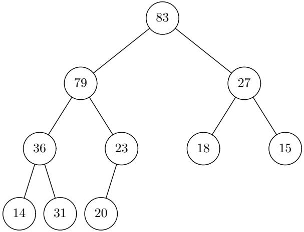

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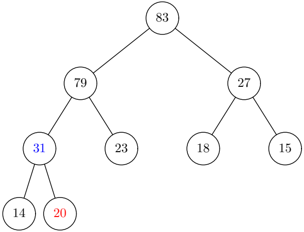

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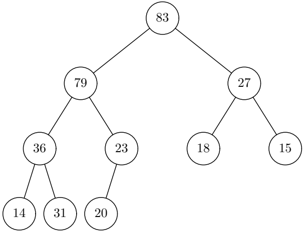

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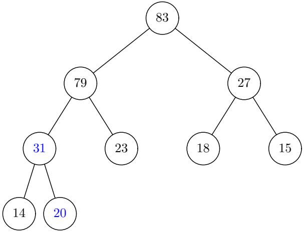

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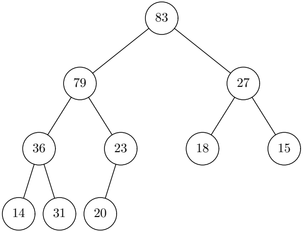

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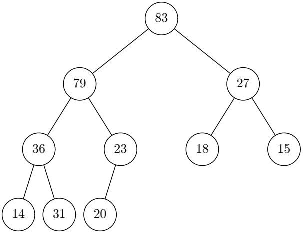

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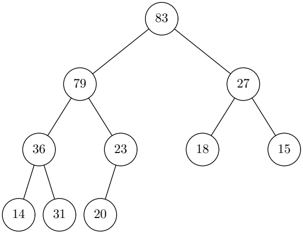

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There is a better way that only requires $O(n)$ time...

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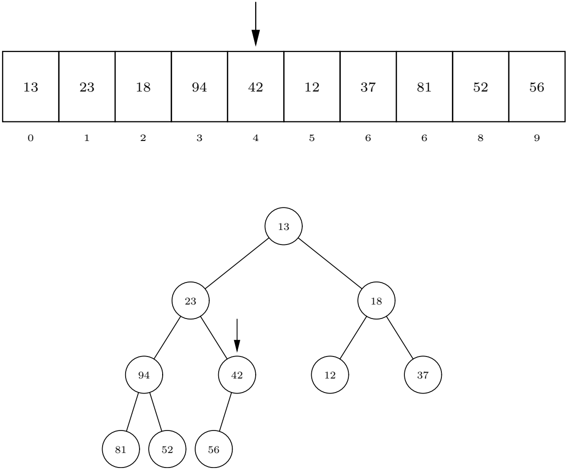
The code given above can be improved: We can start at $i = \lfloor(n\text{-}2)/2\rfloor$ (or equivalently, $i = \lfloor n/2 \rfloor - 1$), rather than $i = n - 1$.

Heapify example

13 23 18 94 42 12 37 81 52 56



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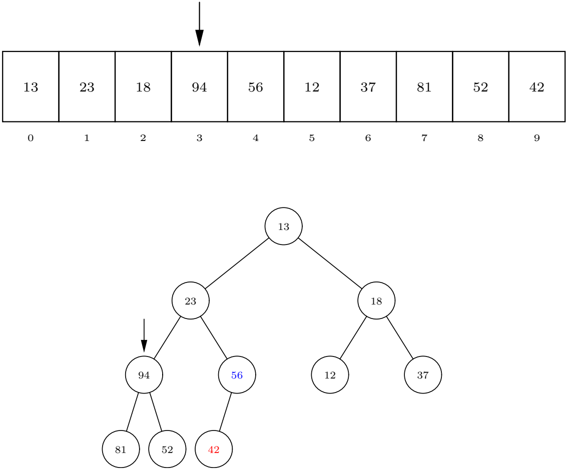

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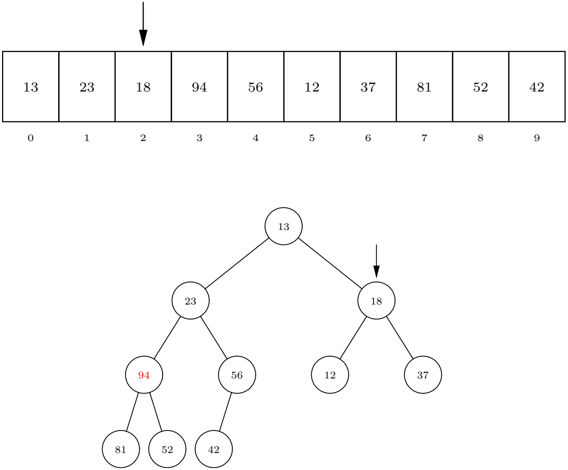

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13	23	37	94	56	12	18	81	52	42
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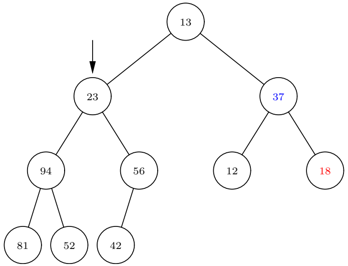

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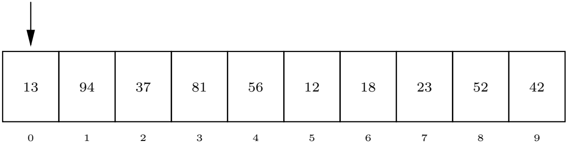

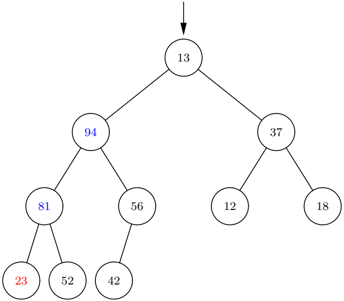

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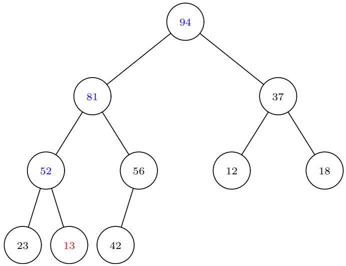

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- Running time: `Heapify` runs in $O(n)$ time. We will prove this on the next slide.

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So heap can be constructed using $O(n)$ comparisons.

Heapsort: version based on Max Sort

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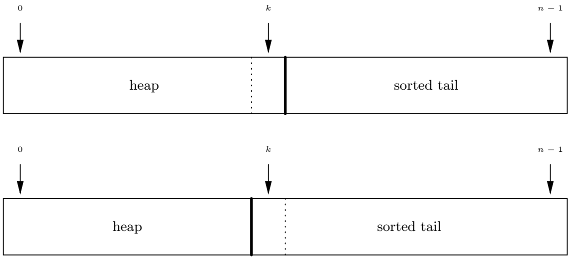
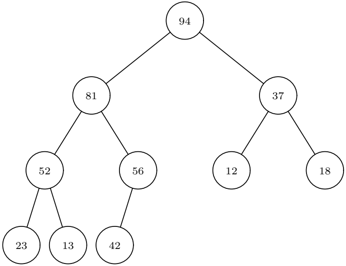


Heapsort example

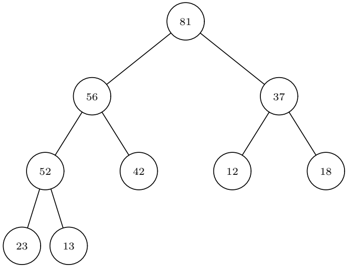
Sort: 13 23 18 94 42 12 37 81 52 56

Heapify:



0	1	2	3	4	5	6	7	8	9
94	81	37	52	56	12	18	23	13	42

Heapsort example, continued



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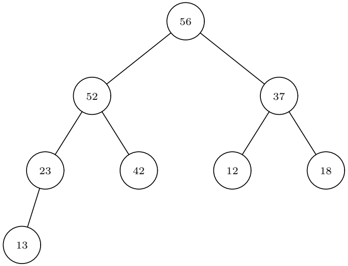

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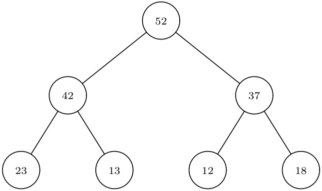

0	1	2	3	4	5	6	7	8	9
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Exercise: Finish this example.

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  - Hence total time is $O(n \log n)$.

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- If we stop after computing the first k entries, total work is

$$O(n + k \log n)$$

Comparison-based sorts: Summary/Comparison

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Sort	Worst-case Time	Storage Requirement	Remarks
Insertion Sort	$O(n^2)$	In-place	Good if input is almost sorted.
QuickSort	$O(n^2)$	$O(\log n)$ extra for stack	$O(n \log n)$ expected time.
Mergesort	$O(n \log n)$	$O(n)$ extra for merge	
Heapsort	$O(n \log n)$	In-place	Can output k smallest in sorted order in $O(n + k \log n)$ time.

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Sort	Stable (without special care)?
Insertion Sort	Yes
Quick-Sort	No
Merge-Sort	Yes (as described here)
Heap-Sort	No

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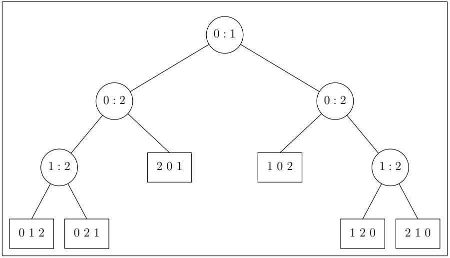

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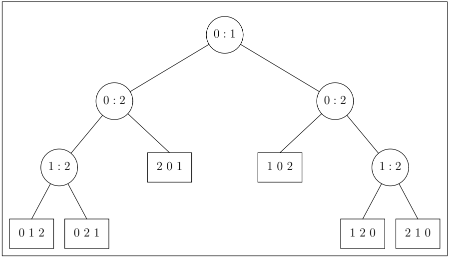

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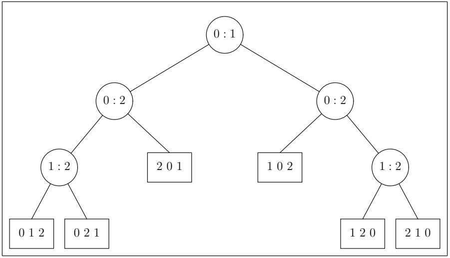

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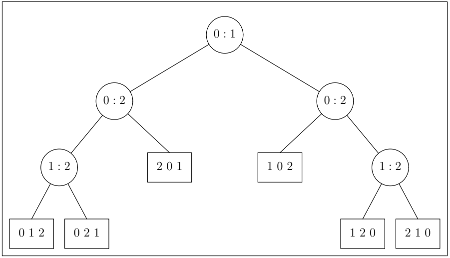

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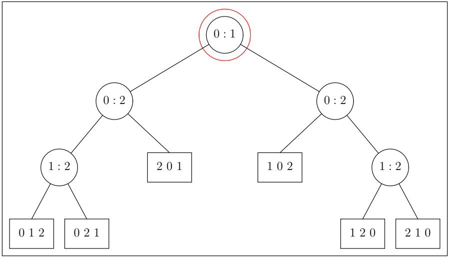

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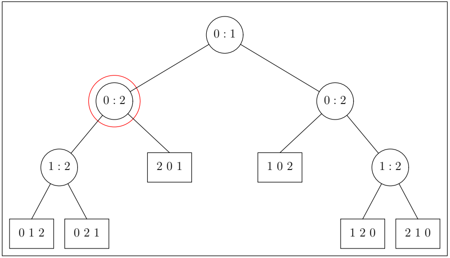

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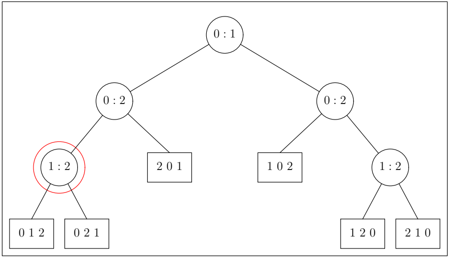

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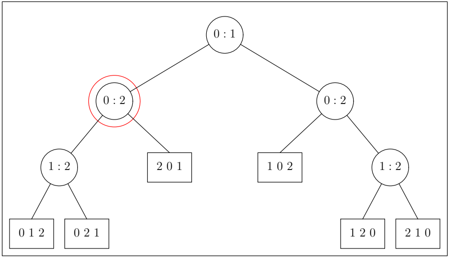

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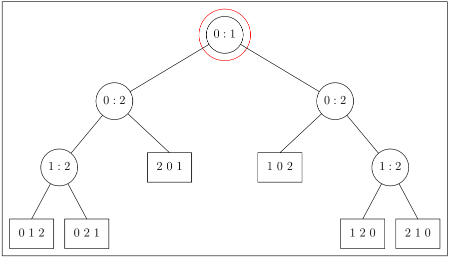

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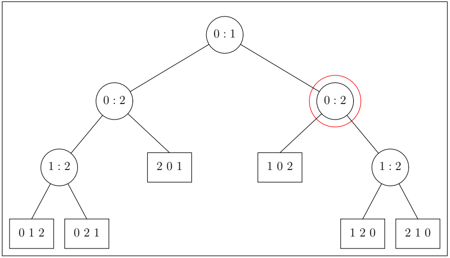

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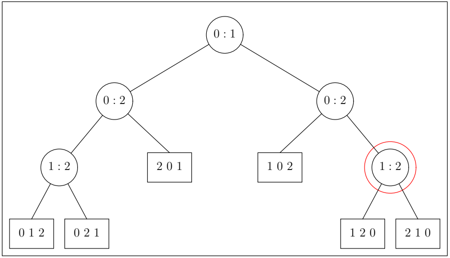

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